ISRAEL JOURNAL OF MATHEMATICS, Vol. 47, Nos. 2-3, 1984

ON TSIRELSON'S SPACE

BY
P. G. CASAZZA,[†] W. B. JOHNSON[††] AND L. TZAFRIRI[††]

ABSTRACT
A structure theory is developed for Tsirelson's example of a Banach space which contains no isomorphic copy of l_p or c_0. In particular, it is shown that this space is the first example, other than subspaces of l_p and c_0, of a Banach space which embeds isomorphically into each of its infinite dimensional subspaces.

Introduction

In this paper we present some additional properties of the space introduced by Tsirelson in [11] and its variations discussed in [5], [6] and [3]. Further results on this interesting space will be presented in [4] and [2].

Tsirelson's original space (denoted here by T^*) was the first example of a Banach space which contains no subspace isomorphic to c_0 or l_p ; $1 \leqq p < \infty$. The notation T for the dual of Tsirelson's original space as well as the analytic description of the norm in T were given in [5]. The convenience of working with a concrete formula for the norm made us prefer this particular notation.

The main result in this paper asserts that T^* is *minimal* in the sense that every infinite dimensional subspace of T^* contains in turn an isomorphic copy of T^*. (In fact, Theorem 14 says that T^* embeds isomorphically into every infinite dimensional subspace of a quotient of T^*.) Previously, only the spaces l_p ; $1 \leqq p < \infty$, c_0 and their infinite dimensional subspaces were known to be minimal.

The minimality of T^* is a consequence of a simple criterion for the boundedness of operators on T and T^* (Theorem 8) and an analysis of the structure of subspaces of quotients of T and T^* which also yields, among other results, the fact that if Y is a quotient space of T or of T^* then every infinite dimensional subspace of Y contains in turn a subspace E which is com-

[†] Supported in part by NSF MCS-8002221 and MCS-8102238.
[††] Supported in part by NSF MCS-7903042.
Received August 3, 1983

plemented in Y and which is isomorphic to the closed linear space of some subsequence of the unit vector basis of T, respectively of T^*.

The fact that T^* is minimal suggested that T^* might also be a prime space (i.e. a space whose infinite dimensional complemented subspaces are all isomorphic to the whole space). This conjecture is disproved in [4]. The main result of [4] is the surprising assertion that T is equal to the so-called modified Tsirelson space T_M, defined in [6]. The fact that T_M, and hence also T^*_M, is not even primary is relatively easy to check (a space X is said to be primary provided that, whenever $X = Y \bigoplus Z$ is a direct sum decomposition, then either Y or Z is isomorphic to X).

Many other results about the structure of T, T^* and related spaces are scattered throughout the paper. For example, it is proved in Theorem 16 that every quotient space of T^* embeds into T^*.

From a technical point of view, one interesting aspect of the present work is that blocking methods, which have been previously used to analyze the structure of l_p and c_0, work equally well for spaces of Tsirelson type. We say that a finite dimensional decomposition (F.D.D., in short) $\{C_n\}_{n=1}^{\infty}$ for a Banach space X is of *type* T (respectively, of *type* T^*) if there are natural numbers $1 \leqq k_1 < k_2 < \cdots$ so that for any x_n in C_n, $n = 1, 2, 3, \cdots$, $\sum_{n=1}^{\infty} x_n$ converges in X if and only if $\sum_{n=1}^{\infty} \| x_n \| t_{k_n}$ converges in T (respectively, $\sum_{n=1}^{\infty} \| x_n \| t^*_{k_n}$ converges in T^*), where $\{t_n\}_{n=1}^{\infty}$ (respectively, $\{t^*_n\}_{n=1}^{\infty}$) is the unit vector basis for T (respectively, T^*). If it is important to specify the sequence $\{k_n\}_{n=1}^{\infty}$, we say that $\{C_n\}_{n=1}^{\infty}$ is of type T (or type T^*) relative to $\{k_n\}_{n=1}^{\infty}$. We show that every F.D.D. for a subspace of a quotient of T (respectively, T^*) has a blocking which is of type T (respectively, type T^*).

We use standard Banach space theory terminology, as may be found in the book [10]. In particular, we refer to [10] for a discussion of F.D.D.'s and blockings thereof.

Results on the Tsirelson space

Throughout this paper we write $E < F$ or $E \leqq F$ when E and F are subsets of the integers which satisfy $\max E < \min F$; respectively, $\max E \leqq \min F$. In case one of the sets, say E, consists of only one non-negative integer n we put $n < F$ or $n \leqq F$ instead of $\{n\} < F$; respectively, $\{n\} \leqq F$.

Let T_0 be the linear space of all sequences of scalars which are eventually zero and let $\{t_n\}_{n=1}^{\infty}$ denote the unit vector basis in T_0. For a vector $x = \sum_{n=1}^{\infty} a_n t_n \in T_0$ and $1 \leqq E$ we define $Ex = \sum_{n \in E} a_n t_n$. Set

$$\| x \|_0 = \max_n | a_n |$$

and, for $m \geqq 0$,

$$\| x \|_{m+1} = \max \left\{ \| x \|_m, 2^{-1} \max \left[\sum_{j=1}^{k} \| E_j x \|_m \right] \right\},$$

where the inner maximum is taken over all choices of finite subsets $\{E_j\}_{j=1}^k$ of the integers so that

$$k \leqq E_1 < E_2 < \cdots < E_k,$$

$k = 1, 2, \cdots$. Any expression of the form $2^{-1} \sum_{j=1}^k \| E_j x \|_m$, with the $\{E_j\}_{j=1}^k$ chosen as above, is called an *admissible sum* for the vector x. The definition of admissible sums presented here differs slightly from that given in [5] by allowing $k \leqq E_1$ instead of $k < E_1$. It will be shown later that both definitions produce equivalent norms; however, the one used here simplifies the notations. It is easily seen that the sequence $\{t_n\}_{n=1}^\infty$ forms an unconditional basis in the completion T of T_0 with respect to the norm

$$\| x \| = \lim_{m \to \infty} \| x \|_m ; \qquad x \in T_0.$$

It is immediately verified that the norm of a vector $x \in T$ satisfies

$$(*) \qquad \| x \| = \sup \left\{ \| x \|_0, 2^{-1} \sup \left[\sum_{j=1}^{k} \| E_j x \| \right] \right\},$$

where the inner supremum is taken again over all choices of finite sets $\{E_j\}_{j=1}^k$ such that $k \leqq E_1 < E_2 < \cdots < E_k$, $k = 1, 2, \cdots$.

It is proved in [11] (see also [5]) that T contains no subspace isomorphic to c_0 or to l_p; $1 \leqq p < \infty$. In particular, T is reflexive.

Our first result explains the way in which the $m+1$-norm of a vector $x \in T$ is actually computed.

PROPOSITION 1. *For any vector $x \in T$ and any integer m, either*

$$\| x \|_{m+1} = \sup \left\{ 2^{-1} \sum_{j=1}^{k} \| E_j x \|_m ;\ k \leqq E_1 < E_2 < \cdots < E_k,\ k = 1, 2, \cdots \right\}$$

or

$$\| x \|_{m+1} = \| x \|_0.$$

PROOF. Suppose that for some vector $x \in T$ and some integer m,

$$\|x\|_{m+1} > \sup\left\{2^{-1}\sum_{j=1}^{k}\|E_jx\|_m\,;\ k \leqq E_1 < E_2 < \cdots < E_k\,;\ k = 1,2,\cdots\right\}.$$

Then, by the definition of $\|\cdot\|_{m+1}$, we have $\|x\|_{m+1} = \|x\|_m$. It follows that

$$\|x\|_m > \sup\left\{2^{-1}\sum_{j=1}^{k}\|E_jx\|_m\,;\ k \leqq E_1 < E_2 < \cdots < E_k\,;\ k = 1,2,\cdots\right\}$$

$$\geqq \sup\left\{2^{-1}\sum_{j=1}^{k}\|E_jx\|_{m-1};\ k \leqq E_1 < E_2 < \cdots < E_k\,;\ k = 1,2,\cdots\right\}$$

and, hence, $\|x\|_m = \|x\|_{m-1}$. Continuing so, we easily conclude that $\|x\|_{m+1} = \|x\|_0$. □

We present now a lemma which is very useful in the study of block basic sequences of $\{t_n\}_{n=1}^{\infty}$.

LEMMA 2. *Fix an integer m and an increasing sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers. Then, for each vector $x = \sum_{n=1}^{\infty} a_n t_{k_n} \in T_0$ there exist finite subsets $\{E_l\}_{l=1}^{3}$ of the integers so that*

$$1 \leqq E_1 < E_2 < E_3$$

and

$$\left\|\sum_{n=1}^{\infty} a_n t_{k_{2n}}\right\|_m \leqq \sum_{l=1}^{3}\|E_lx\|_m.$$

PROOF. The proof is done by induction on m where the case $m = 0$ is obvious. Suppose now that the assertion is correct for some m and all the vectors $x \in T_0$. Fix a vector $x = \sum_{n=1}^{\infty} a_n t_{k_n}$ and put $y = \sum_{n=1}^{\infty} a_n t_{k_{2n}}$. If $\|y\|_{m+1} = \|y\|_m$ the case $m+1$ is completed immediately. Thus, we can assume the existence of a sequence $\{F_j\}_{j=1}^{k_{2n}}$ of finite disjoint subsets of the integers such that

$$k_{2n} \leqq F_1 < F_2 < \cdots < F_{k_{2n}},$$

and

$$\|y\|_{m+1} = 2^{-1}\sum_{j=1}^{k_{2n}}\|F_jy\|_m.$$

By applying the induction hypothesis to each of the vectors F_jy; $j = 1,2,\cdots,k_{2n}$, we can find sets $\{E_j^l\}_{j=1,\,l=1}^{k_{2n},\ 3}$ so that

$$\|y\|_{m+1} \leqq 2^{-1}\sum_{j=1}^{k_{2n}}\sum_{l=1}^{3}\|E_j^lx\|_m.$$

Notice that there is no loss of generality in assuming that the set $\{E_j^l\}_{j=1,\,l=1}^{k_{2n},\ 3}$ are mutually disjoint and that $E_1^l \geqq k_n$ for $l = 1,2,3$. Hence, by reindexing these sets as $\{G_h\}_{h=1}^{3k_{2n}}$, we have

$$k_n \leqq G_1 < G_2 < \cdots < G_{3k_{2n}}$$

and

$$\begin{aligned}\|y\|_{m+1} &\leqq 2^{-1}\sum_{h=1}^{3k_{2n}} \|G_h x\|_m \\ &= 2^{-1}\sum_{h=1}^{k_n}\|G_h x\|_m + 2^{-1}\sum_{h=k_n+1}^{k'_n}\|G_h x\|_m + 2^{-1}\sum_{h=k'_n+1}^{3k_{2n}}\|G_h x\|_m,\end{aligned}$$

where $k'_n = k_n + k_{n+k_n}$. Let

$$E_1 = \bigcup_{h=1}^{k_n} G_h, \quad E_2 = \bigcup_{h=k_n+1}^{k'_n} G_h \quad \text{and} \quad E_3 = \bigcup_{h=k'_n+1}^{3k_{2n}} G_h.$$

Since $k_n \leqq G_1$ it follows that $2^{-1}\sum_{h=1}^{k'_n}\|G_h x\|_m$ is an admissible sum for the vector $E_1 x$. Since there is no loss of generality in assuming that $(\min G_h)x \neq 0$ for all h it follows that $k'_n - k_n = k_{n+k_n} \leqq G_{k_n+1}$; i.e., also the sum $2^{-1}\sum_{h=k_n+1}^{k'_n}\|G_h x\|_m$ is admissible for the vector $E_2 x$.

In a similar manner we verify that

$$k_{n+k_n+k'_n} \leqq G_{k'_n+1} < \cdots < G_{3k_{2n}}.$$

Hence, it remains to check that $3k_{2n} - k'_n \leqq k_{n+k'_n}$ in order to conclude that also $2^{-1}\sum_{h=k'_n+1}^{3k_{2n}}\|G_h x\|_m$ is an admissible sum for $E_3 x$, and thus obtain

$$\|y\|_{m+1} \leqq \sum_{l=1}^{3}\|E_l x\|_{m+1}.$$

Indeed, this follows by adding the following two inequalities:

$$k_{2n} \leqq k_{n+k_n} = k'_n - k_n \leqq k'_n$$

and

$$2k_{2n} \leqq 2k_{n+k_n} \leqq k_{n+k_n+k_{n+k_n}} = k_{n+k'_n}. \qquad \square$$

It is clear from the definition of the norm in T that, whenever $\{k_n\}_{n=1}^\infty$ and $\{j_n\}_{n=1}^\infty$ are two increasing sequences of integers so that $k_n \leqq j_n$, for all n, then

$$\left\|\sum_{n=1}^{\infty} a_n t_{k_n}\right\| \leqq \left\|\sum_{n=1}^{\infty} a_n t_{j_n}\right\|,$$

for all choices of scalars $\{a_n\}_{n=1}^{\infty}$. By using this observation and Lemma 2, we get immediately the following result:

PROPOSITION 3. *For every increasing sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ and any choice of scalars $\{a_n\}_{n=1}^{\infty}$, we have*

$$\left\| \sum_{n=1}^{\infty} a_n t_{k_n} \right\| \leqq \left\| \sum_{n=1}^{\infty} a_n t_{k_{2n}} \right\| \leqq 3 \left\| \sum_{n=1}^{\infty} a_n t_{k_n} \right\| .$$

REMARK. Proposition 6 contains actually a much stronger result than Proposition 3, but Proposition 6 requires Proposition 3 for its proof.

It follows from Proposition 3 that every subsequence of $\{t_n\}_{n=1}^{\infty}$ spans a Banach space which is isomorphic to its cartesian square.

We present now some results concerning block basic sequences in T.

LEMMA 4. *Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i$; $n = 1, 2, \cdots$ be a normalized block basic sequence of $\{t_n\}_{n=1}^{\infty}$. Then, for any sequence of scalars $\{b_n\}_{n=1}^{\infty}$, we have*

$$\left\| \sum_{n=1}^{\infty} b_n t_{p_n+1} \right\| \leqq \left\| \sum_{n=1}^{\infty} b_n y_n \right\| .$$

PROOF. We shall prove by induction on m that

$$\left\| \sum_{n=1}^{\infty} b_n t_{p_n+1} \right\|_m \leqq \left\| \sum_{n=1}^{\infty} b_n y_n \right\| ,$$

for every choice of scalars $\{b_n\}_{n=1}^{\infty}$. The case $m = 0$ is trivial since $\| y_n \| = 1$ for all n. Suppose now that the inequality holds for some m and all $\{b_n\}_{n=1}^{\infty}$. Fix n and take a sequence $\{E_j\}_{j=1}^{n}$ of finite subsets of the integers such that

$$n \leqq E_1 < E_2 < \cdots < E_n .$$

Then, without loss of generality we may assume that each set E_j is an interval of the form $[p_k + 1, p_l]$. Now, choose scalars $\{b_n\}_{n=1}^{\infty}$ and put $x = \sum_{n=1}^{\infty} b_n t_{p_n+1}$ and $y = \sum_{n=1}^{\infty} b_n y_n$. By using the induction hypothesis for each of the vectors $\{E_j y\}_{j=1}^{n}$ it follows that

$$2^{-1} \sum_{j=1}^{n} \| E_j x \|_m \leqq 2^{-1} \sum_{j=1}^{n} \| E_j y \| \leqq \| y \| .$$

This, of course, completes the proof. □

LEMMA 5. *Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i$; $n = 1, 2, \cdots$ be a normalized block basic sequence of $\{t_n\}_{n=1}^{\infty}$. Then, for any sequence of scalars $\{b_n\}_{n=1}^{\infty}$, we have*

$$\left\| \sum_{n=1}^{\infty} b_n y_n \right\| \leqq 6 \left\| \sum_{n=1}^{\infty} b_n t_{p_{n+1}} \right\| .$$

The proof of Lemma 5 will require the introduction of a new norm on T_0. The norm $|||\cdot|||$ is defined exactly as $\|\cdot\|$ except that in the inner maximum of

$$|||x|||_{m+1}=\max\left\{|||x|||_m, 2^{-1}\max\left[\sum_{j=1}^{2k}|||E_jx|||_m\right]\right\}$$

we allow the finite subsets $\{E_j\}_{j=1}^{2k}$ to satisfy

$$k\leqq E_1<E_2<\cdots<E_{2k}.$$

It is easily verified that

$$\left|\left|\left|\sum_{n=1}^{\infty}c_nt_n\right|\right|\right|=\left\|\sum_{n=1}^{\infty}c_nt_{2n}\right\|,$$

for every choice of scalars $\{c_n\}_{n=1}^{\infty}$ which is eventually zero. It follows from Proposition 3 that

$$\|x\|\leqq|||x|||\leqq 3\|x\|,$$

for any $x\in T_0$ and thus also for any vector $x\in T$.

We can return now to the proof of Lemma 5. In view of the remark made above, it clearly suffices to prove that, for any choice of scalars $\{b_n\}_{n=1}^{\infty}$ and any integer m, we have

$$\left\|\sum_{n=1}^{\infty}b_ny_n\right\|_m\leqq 2\left|\left|\left|\sum_{n=1}^{\infty}b_nt_{p_n+1}\right|\right|\right|.$$

The case $m=0$ is trivial so we will assume that this inequality holds for some m and all choices of scalars $\{b_n\}_{n=1}^{\infty}$. Put $x=\sum_{n=1}^{\infty}b_nt_{p_n+1}$ and $y=\sum_{n=1}^{\infty}b_ny_n$, for some choice of scalars $\{b_n\}_{n=1}^{\infty}$ such that both series converge. Select an integer k and a sequence $\{E_j\}_{j=1}^{k}$ of finite intervals of the integers so that $k\leqq E_1<E_2<\cdots<E_k$. By enlarging the set E_k if needed, we can assume without loss of generality that $\max E_k=p_{j+1}$, for some $j>k$. For each j and n, let $F_n=\{p_n+1,p_n+2,\cdots,p_{n+1}\}$ and $G_j=\bigcup\{F_n\,;F_n\subset E_j\}$. Put

$$H=\{n\,;F_n\cap E_j\neq\emptyset\text{ for some }1\leqq j\leqq k\text{ but }F_n\not\subset E_j\}$$

and notice that $\bar{\bar{H}}\leqq k$. Now, by the induction hypothesis and the definition of the norms $\|\cdot\|$ and $|||\cdot|||$, we get

$$2^{-1}\sum_{j=1}^{k}\|E_jy\|_m\leqq 2^{-1}\sum_{j=1}^{k}\|G_jy\|_m+2^{-1}\sum_{j=1}^{k}\sum_{l\in H}\|(E_j\cap F_l)y\|_m$$

$$\leqq 2^{-1}\sum_{j=1}^{k}2|||G_jx|||+2^{-1}\sum_{l\in H}2\|F_ly\|_m\leqq\sum_{j=1}^{k}|||G_jx|||+\sum_{l\in H}\|F_ly\|,$$

since $\|u\|_m \leqq \|u\| \leqq |||u|||$, for all $u \in T$. But, for each $l \in H$,

$$\|F_l y\| = \|b_l y_l\| = |b_l|.$$

Hence,

$$2^{-1}\sum_{j=1}^{k}\|E_j y\|_m \leqq \sum_{j=1}^{k}|||G_j x||| + \sum_{l\in H}|||b_l t_{p_l+1}|||.$$

However, the expression appearing on the right-hand side contains at most $2k$ elements since $\bar{\bar{H}} \leqq k$. It follows that this is twice an admissible sum for x relative to the norm $|||\cdot|||$ and thus

$$2^{-1}\sum_{j=1}^{k}\|E_j y\|_m \leqq 2|||x|||.$$

This, of course, implies that $\|y\|_{m+1} \leqq 2|||x|||$. □

As an immediate consequence of Proposition 3 and Lemmas 4 and 5, we obtain the following result:

PROPOSITION 6. *Let* $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i$; $n = 1, 2, \cdots$ *be a normalized block basic sequence of* $\{t_n\}_{n=1}^{\infty}$ *in* T. *Then, for any choice of integers* $p_n < k_n \leqq p_{n+1}$; $n = 1, 2, \cdots$ *and scalars* $\{b_n\}_{n=1}^{\infty}$, *we have*

$$3^{-1}\left\|\sum_{n=1}^{\infty}\|E_n x\|\, t_{k_n}\right\| \leqq \left\|\sum_{n=1}^{\infty} b_n y_n\right\| \leqq 18\left\|\sum_{n=1}^{\infty} b_n t_{k_n}\right\|.$$

COROLLARY 7. (i) *Let* $\{E_n\}_{n=1}^{\infty}$ *be a sequence of finite subsets of the integers such that* $1 \leqq E_1 < E_2 < \cdots$ *and let* $k_n \in E_n$ *for all* n. *Then for each* x *in* T,

$$3^{-1}\left\|\sum_{n=1}^{\infty}\|E_n x\|\, t_{k_n}\right\| \leqq \left\|\sum_{n=1}^{\infty} E_n x\right\| \leqq 18\left\|\sum_{n=1}^{\infty}\|E_n x\|\, t_{k_n}\right\|.$$

(ii) *More generally, let* $\{A_i\}_{i=1}^{\infty}$ *be an F.D.D. of type* T *relative to* $\{p(i)\}_{i=1}^{\infty}$, *let* $B_n = [A_i]_{i=l_n}^{l_{n+1}-1}$ *be a blocking of* $\{A_i\}_{i=1}^{\infty}$, *and let* $p(l_n) \leqq k_n \leqq p(l_{n+1}-1)$ *for all* n. *Then* $\{B_n\}_{n=1}^{\infty}$ *is an F.D.D. of type* T *relative to* $\{k_n\}_{n=1}^{\infty}$.

PROOF. To prove (i), simply apply Proposition 6 to the block basic sequence $\{E_n x/\|E_n x\|\}_{n=1}^{\infty}$, $E_n x \neq 0$.

To prove (ii), assume, without loss of generality, that for all x_i in A_i the norm is given by

$$\left\|\sum_{i=1}^{\infty} x_i\right\| = \left\|\sum_{i=1}^{\infty}\|x_i\|\, t_{p(i)}\right\|.$$

In particular, for $0 \neq y_n$ in B_n,

$$y_n = \sum_{i=l_n}^{l_{n+1}-1} x_i \qquad \text{with } x_i \in A_i,$$

we have that

$$\| y_n \| = \left\| \sum_{i=l_n}^{l_{n+1}-1} \| x_i \| t_{p(i)} \right\| .$$

Proposition 6 thus yields for such y_n's that

$$3^{-1} \left\| \sum_{n=1}^{\infty} \| y_n \| t_{k_n} \right\| \leqq \left\| \sum_{n=1}^{\infty} \| y_n \| \left(\| y_n \|^{-1} \sum_{i=l_n}^{l_{n+1}-1} \| x_i \| t_{p(i)} \right) \right\|$$

$$\leqq 18 \left\| \sum_{n=1}^{\infty} \| y_n \| t_{k_n} \right\|$$

and the middle term in the above inequality is equal to $\| \Sigma_{i=1}^{\infty} x_i \|$; i.e., to $\| \Sigma_{n=1}^{\infty} y_n \|$. □

Our next result provides a basic tool for the construction of bounded linear operators on spaces which have an F.D.D. of type T.

THEOREM 8. *Let $\{U_n\}_{n=1}^{\infty}$ be an F.D.D. for X of type T relative to $\{k_n\}_{n=1}^{\infty}$, let $\{V_n\}_{n=1}^{\infty}$ be an F.D.D. for Y of type T relative to $\{j_n\}_{n=1}^{\infty}$, and assume that for some i and all n,*

$$k_n \leqq j_{n+i} \quad \textit{and} \quad j_n \leqq k_{n+i}.$$

Suppose that $L_n : U_n \to V_n$ are linear operators with $\sup_n \| L_n \| < \infty$. Define (formally) $L : X \to Y$ by

$$L\left(\sum_{n=1}^{\infty} x_n \right) = \sum_{n=1}^{\infty} L_n x_n \, ; \qquad x_n \in U_n, \quad n = 1, 2, \cdots .$$

Then L is bounded. Moreover, if in addition each L_n is one-to-one and $\sup_n \| L_n^{-1} \| < \infty$, then L is an isomorphism from X into Y.

PROOF. This theorem reduces to the obvious case where $j_n = k_n$ for all n, because Proposition 6 yields that $\{t_{j_n}\}_{n=1}^{\infty}$ is equivalent to $\{t_{k_n}\}_{n=1}^{\infty}$, which implies that $\{V_n\}_{n=1}^{\infty}$ is an F.D.D. of type T relative to $\{k_n\}_{n=1}^{\infty}$. □

REMARK. Notice that if $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ are two sequences of finite subsets of the integers so that $1 \leqq E_1 < E_2 < \cdots$, $1 \leqq F_1 < F_2 < \cdots$, and $E_{n-1} < F_n < E_{n+2}$ for all n, then Theorem 8 applies to the F.D.D.'s $U_n = [t_i]_{i \in E_n}$, $V_n = [t_i]_{i \in F_n}$; $n = 1, 2, \cdots$, for T.

COROLLARY 9. *Every block basic sequence of $\{t_n\}_{n=1}^{\infty}$ spans a complemented subspace of T.*

PROOF. Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i \neq 0$; $n = 1, 2, \cdots$, be a block basic sequence of $\{t_n\}_{n=1}^{\infty}$. For each n, there is a norm one projection P_n from $[t_i]_{i=p_n+1}^{p_{n+1}}$ onto the one dimensional subspace generated by y_n. Let $E_n = \{p_n+1, p_n+2, \cdots, p_{n+1}\}$; $n = 1, 2, \cdots$. Then, by Theorem 8, the operator $P: T \to T$, defined by $Px = \sum_{n=1}^{\infty} P_n(E_n x)$; $x \in T$, is a bounded linear projection onto $[y_n]_{n=1}^{\infty}$. □

THEOREM 10. *Let $\{k_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers and, for each n, put $W_n = [t_i]_{i=k_n+1}^{k_{n+1}}$. Then, the subsequence $\{t_{k_n}\}_{n=1}^{\infty}$ is equivalent to $\{t_n\}_{n+1}^{\infty}$ if and only if*

$$\sup_n \| I_n \| < \infty,$$

where I_n is the formal identity map from W_n into l_1.

PROOF. We shall assume first that $\{t_{k_n}\}_{n=1}^{\infty}$ is M-equivalent to $\{t_n\}_{n=1}^{\infty}$, for some $m \geqq 1$. Fix n and consider a vector of the form $u_n = \sum_{i=k_n+1}^{k_{n+1}} a_i t_i$. Since $k_{n+1} - k_n \leqq k_{k_n+1}$ it follows that $2^{-1} \sum_{i=k_n+1}^{k_{n+1}} |a_i|$ is an admissible sum for the vector $v_n = \sum_{i=k_n+1}^{k_{n+1}} a_i t_{k_i}$. Hence,

$$2^{-1} \sum_{i=k_n+1}^{k_{n+1}} |a_i| \leqq \| v_n \| \leqq M \| u_n \|,$$

which, in view of the remark made above, implies that $\| I_n \| \leqq 2M$, for all n.

In order to prove the converse, we shall assume that

$$\sup_n \| I_n \| = C < \infty.$$

Let $j(0) = 0$ and, for $n \geqq 1$, let $j(n) = k_{j(n-1)+1}$. This definition ensures that $\{j(n)\}_{n=0}^{\infty}$ is an increasing sequence. Now, for each n, put

$$E_n = \{i; j(n-1) < i \leqq j(n)\} \quad \text{and} \quad F_n = \{k_i; i \in E_n\}.$$

Since $\max E_n = j(n)$, $\min E_n = j(n-1)+1$, $\max F_n = k_{j(n)}$ and $\min F_n = k_{j(n-1)+1} = j(n)$, we have

$$E_{n-1} \leqq j(n-1) < F_n < k_{j(n)+1} = j(n+1) < E_{n+2},$$

for all n; i.e., all the conditions imposed on $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ in the remark after Theorem 8 are satisfied. Put $U_n = [t_i]_{i \in E_n}$ and $V_n = [t_i]_{i \in F_n}$, and define an operator $L_n : U_n \to V_n$ by

$$L_n\left(\sum_{i\in E_n} a_i t_i\right) = \sum_{i\in E_n} a_i t_{k_i},$$

for all choices of scalars $\{a_i\}_{i\in E_n}$. Obviously, the operator $L : T \to T$, defined formally by $Lx = \sum_{n=1}^{\infty} L_n(E_n x)$; $x \in T$, has the property that it maps t_n into t_{k_n}, for all n (since $\bigcup_{n=1}^{\infty} E_n$ consists of all the positive integers). Consequently, the proof of Theorem 10 will be completed once we show that L is bounded or, in view of the remark following Theorem 8, that $\sup_n \|L_n\| < \infty$.

For every h, put $A_h = \{1 + k_{j(n-2)+h}, 2 + k_{j(n-2)+h}, \cdots, k_{j(n-2)+h+1}\}$. Then, for every $n > 1$ and every sequence $\{a_i\}_{i\in E_n}$, we have, by Lemma 4, that

$$\left\|\sum_{i\in E_n} a_i t_i\right\| = \left\|\sum_{h=1}^{j(n-1)-j(n-2)} \sum_{i\in A_h} a_i t_i\right\|$$

$$\geqq \left\|\sum_{h=1}^{j(n-1)-j(n-2)} \left\|\sum_{i\in A_h} a_i t_i\right\| t_{1+k_{j(n-2)+h}}\right\|.$$

However, our assumption on I_n implies that

$$\sum_{i\in A_h} |a_i| \leqq C \left\|\sum_{i\in E_n} a_i t_{k_i}\right\|.$$

Hence,

$$\left\|\sum_{i\in E_n} a_i t_i\right\| \geqq C^{-1} \left\|\sum_{h=1}^{j(n-1)-j(n-2)} \left(\sum_{i\in A_h} |a_i|\right) t_{1+k_{j(n-2)+h}}\right\|.$$

On the other hand, it is readily verified that the expression $2^{-1}\sum_{h=1}^{j(n-1)-j(n-2)}(\sum_{i\in A_h}|a_i|)$ is an admissible sum for the vector $\sum_{h=1}^{j(n-1)-j(n-2)}(\sum_{i\in A_h}|a_i|)t_{1+k_{j(n-2)+h}}$. Thus,

$$\left\|\sum_{i\in E_n} a_i t_i\right\| \geqq 2^{-1}C^{-1} \sum_{i\in E_n} |a_i| \geqq 2^{-1}C^{-1} \left\|\sum_{i\in E_n} a_i t_{k_i}\right\|.$$

It follows that $\|L_n\| \leqq 2C$ for all $n > 1$ and, since L_1 is clearly a bounded operator, this concludes the proof. □

REMARK. Theorem 10 was formulated and proved for a subsequence $\{t_{k_n}\}_{n=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$. Sometimes, it is necessary to be able to compare two general subsequences $\{t_{k_n}\}_{n=1}^{\infty}$ and $\{t_{j_n}\}_{n=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$.

Put $M_i = \{k_n ; j_{i-1} < k_n \leqq j_i\}$ and $N_i = \{j_n ; k_{i-1} < j_n \leqq k_i\}$, for $i = 1, 2, \cdots$. Thus, one can show, by exactly the same methods as those used in the proof of Theorem 10, that $\{t_{k_n}\}_{n=1}^{\infty}$ is equivalent to $\{t_{j_n}\}_{n=1}^{\infty}$ if and only if

$$\sup_i \max\{\|I_i\|, \|J_i\|\} < \infty$$

where I_i (respectively, J_i) is the formal identity map from $[t_{j_n}]_{n\in N_i}$ (respectively, $[t_{k_n}]_{n\in M_i}$) into l_1.

In the case when $\{t_{j_n}\}_{n=1}^\infty$ is a subsequence of $\{t_{k_n}\}_{n=1}^\infty$ we clearly have $\bar{\bar{N}}_i = 1$, for all i. Therefore, one can conclude that a subsequence $\{t_{j_n}\}_{n=1}^\infty$ of $\{t_{k_n}\}_{n=1}^\infty$ is equivalent to $\{t_{k_n}\}_{n=1}^\infty$ if and only if

$$\sup_i \|J_i\| < \infty.$$

We would like to describe now some concrete cases when lacunary sequences $\{k_n\}_{n=1}^\infty$ have the property that $\{t_{k_n}\}_{n=1}^\infty$ is equivalent to $\{t_n\}_{n=1}^\infty$. To this end we introduce a sequence $\{\varphi_i\}_{i=0}^\infty$ of functions defined on the positive integers by letting:

$$\varphi_0(n) = n2^n, \qquad \text{for all } n,$$

and

$$\varphi_{i+1}(n) = \underbrace{(\varphi_i \circ \varphi_i \circ \cdots \circ \varphi_i)}_{\varphi_i(n)-n \quad \text{iterations}}(n),$$

for $i \geqq 0$ and all n. With this notation, we prove first the following simple lemma:

Lemma 11. *Fix integers i and n, and let $\{E_j\}_{j=n}^k$ be a sequence of non-empty finite subsets of the integers so that*

$$n \leqq E_n < E_{n+1} < \cdots < E_k$$

with $k < \varphi_i(n)$. Then, for each $x \in T$,

$$(*_i) \qquad \|x\| \geqq 4^{-2^i} \sum_{j=n}^{k} \|E_j x\|.$$

Proof. We fix n and $x \in T$, and prove the lemma by induction on i. For the case $i = 0$, we assume for notational convenience that the sequence $\{E_j\}_{j=1}^k$ has the maximal number of elements allowed; i.e., $k = n2^n - 1$. By using $(*)$, we get

$$\|x\| \geqq 2^{-1} \sum_{l=1}^{n} \left\| \sum_{j=n2^{l-1}}^{n2^l-1} E_j x \right\|.$$

Now, notice that for each $1 \leqq l \leqq n$, $n2^{l-1} \leqq E_{n2^{l-1}} < \cdots < E_{n2^l-1}$. Hence, by using again $(*)$, we obtain

$$\left\| \sum_{j=n2^{l-1}}^{n2^l-1} E_j x \right\| \geqq 2^{-1} \sum_{j=n2^{l-1}}^{n2^l-1} \|E_j x\|.$$

Combining these inequalities we finally conclude that

$$\|x\| \geqq 4^{-1} \sum_{j=n}^{n2^n-1} \|E_j x\|,$$

which completes the case $i = 0$.

For the induction step we use exactly the same argument with the exception that instead of $(*)$ we apply $(*_i)$. □

COROLLARY 12. *For each fixed* $i \geqq 0$, *the sequence* $\{t_{\varphi_i(n)}\}_{n=1}^{\infty}$ *is equivalent to* $\{t_n\}_{n=1}^{\infty}$.

PROOF. By Lemma 11 applied to a vector x in T of the form $\sum_{j=\varphi_i(n-1)}^{\varphi_i(n)-1} a_j t_j$ with $E_j = \{j\}$ for $\varphi_i(n-1) \leqq j < \varphi_i(n)$, we easily conclude that $\|I_n\| \leqq 4^{2^i}$ where I_n is the formal identity operator from $[t_j]_{j=\varphi_i(n-1)}^{\varphi_i(n)-1}$ into l_1. This completes the proof in view of Theorem 10. □

REMARKS. (1) The functions $\{\varphi_i(n)\}_{n=1}^{\infty}$; $i = 1, 2, \cdots$ have a very fast rate of growth. One case which illustrates well this situation is the following. Put $\exp_0 n = n$, for all n, and then, for $j, n \geqq 1$, set $\exp_j n = 2^{\exp_{j-1} n}$. It is easily verified that $\exp_n n \leqq \varphi_1(n)$, for all n. Hence, by Corollary 12, $\{t_{\exp_n n}\}_{n=1}^{\infty}$ is equivalent to $\{t_n\}_{n=1}^{\infty}$ itself. This result is used in [2].

(2) Most of the results stated so far in this paper for T or for F.D.D.'s of type T are valid also for T^* or for F.D.D.'s of type T^*, when suitable modifications are made. Generally the "dual" results are deduced from the results we have stated in a purely formal manner.

We shall denote the unit vector basis in T^* by $\{t_n^*\}_{n=1}^{\infty}$. Then, for instance, Proposition 6, adapted for T^*, asserts that if $z_n^* = \sum_{i=q_n+1}^{q_{n+1}} c_i t_i^*$; $n = 1, 2, \cdots$ is a normalized block basic sequence of $\{t_n^*\}_{n=1}^{\infty}$ and $q_n < h_n \leqq q_{n+1}$, for all n, then

$$18^{-1} \left\| \sum_{n=1}^{\infty} d_n t_{h_n}^* \right\| \leqq \left\| \sum_{n=1}^{\infty} d_n z_n^* \right\| \leqq 3 \left\| \sum_{n=1}^{\infty} d_n t_{h_n}^* \right\|,$$

for any choice of $\{d_n\}_{n=1}^{\infty}$. Naturally, also Theorem 8 and the remark following can be extended easily to the case of F.D.D.'s of type T^*. The adaptation of Theorem 8 to T^* will be used in the sequel. As is readily verified, Corollary 9 remains valid in T^*, as stated.

Finally, Theorem 11 remains true also in T^*, in the sense that a subsequence $\{t_{k_n}\}_{n=1}^{\infty}$ is equivalent to $\{t_n^*\}_{n=1}^{\infty}$ if and only if

$$\sup_n \|I_n^{-1}\| < \infty,$$

where I_n is the formal identity from $[t_i^*]_{i=k_n+1}^{k_{n+1}}$ into l_∞. This is checked by duality. Corollary 12 is valid, too.

(3) In [1], Bellenot uses Theorem 10 to give a complete description in terms of the growth rate of $\{k_n\}_{n=1}^\infty$ of when $\{t_{k_n}\}_{n=1}^\infty$ is equivalent to $\{t_n\}_{n=1}^\infty$.

In the last part of the paper we present a blocking principle for F.D.D.'s in subspaces of quotient spaces of spaces having an F.D.D. of type T or type T^*, which has some interesting applications.

THEOREM 13 (the blocking principle). *Let $\{C_n\}_{n=1}^\infty$ be an F.D.D. of a subspace Y of a quotient space of a Banach space X, and assume that X has an F.D.D. $\{A_n\}_{n=1}^\infty$ which is of type T (respectively, of type T^*). Then $\{C_n\}_{n=1}^\infty$ has a blocking $\{D_i\}_{i=1}^\infty$ which is a F.D.D. of type T (respectively, of type T^*).*

PROOF. By duality, it suffices to prove Theorem 13 for T only. Suppose that Y is a subspace of a quotient space Z of X and denote by $Q : X \xrightarrow{\text{onto}} Z$ the quotient map. Let $\{C_n^*\}_{n=1}^\infty$ be the F.D.D. of Y^* determined by $\{C_n\}_{n=1}^\infty$. Let M be the F.D.D. constant of $\{C_n\}_{n=1}^\infty$ in the sense that, whenever $y_n \in C_n$, for all n, and $k < j$, we have

$$\left\| \sum_{n=1}^{k} y_n \right\| \leqq M \left\| \sum_{n=1}^{j} y_n \right\| .$$

The dual Y^* of Y is isometric to a subspace of a quotient space W of X^*. Denote by $R : X^* \xrightarrow{\text{onto}} W$ the corresponding quotient map.

By [7], [8] (see also proposition 1.g.4 (b) from [10] and the remark thereafter) applied successively to Q and R, we can construct blockings

$$B_i = [A_j]_{j=p(i)}^{p(i+1)-1} \quad \text{and} \quad B_i^* = [A_j^*]_{j=p(i)}^{p(i+1)-1}; \qquad i = 1, 2, \cdots$$

of the given F.D.D. for X and the corresponding F.D.D. for X^*, and blockings

$$D_i = [C_j]_{j=q(i)}^{q(i+1)-1} \quad \text{and} \quad D_i^* = [C_j^*]_{j=q(i)}^{q(i+1)-1}; \qquad i = 1, 2, \cdots$$

of the given F.D.D. for Y and the corresponding F.D.D. for Y^* so that:

(1) For each i and each $y \in D_i$, there exists an element $x \in B_i \oplus B_{i+1}$ for which:

$$\| Qx - y \| \leqq M^{-1} 2^{-i-2} \| y \| \quad \text{and} \quad \| x \| \leqq 8 \| y \| .$$

(2) For each i and each $y_i^* \in D_i^*$ there exists an element $x^* \in B_i^* \oplus B_{i+1}$ for which

$$\| Rx^* - y^* \| \leqq M^{-1} 2^{-i-2} \| y^* \| \quad \text{and} \quad \| x^* \| \leqq 8 \| y^* \| .$$

Now, let $y_i \in D_i$; $i = 1,2,\cdots$ be a sequence of vectors which is eventually zero and let $x_i \in B_i \oplus B_{i+1}$ be the corresponding sequence of vectors given by (1). Then,

$$\left\| \sum_{i=1}^{\infty} y_i \right\| \leqq \left\| \sum_{i=1}^{\infty} Qx_i \right\| + \left\| \sum_{i=1}^{\infty} (y_i - Qx_i) \right\|$$

$$\leqq \left\| \sum_{i=1}^{\infty} x_{2i} \right\| + \left\| \sum_{i=1}^{\infty} x_{2i-1} \right\| + M^{-1} \sum_{i=1}^{\infty} 2^{-i-2} \| y_i \|.$$

By Corollary 7 (ii), $\{B_n\}_{n=1}^{\infty}$ is an F.D.D. of type T relative to some sequence $\{k_n\}_{n=1}^{\infty}$. Using this and the fact that $\| y_i \| \leqq 2M \| \Sigma_{k=1}^{\infty} y_k \|$, for all i, we obtain

$$\left\| \sum_{i=1}^{\infty} y_i \right\| \leqq K \left\| \sum_{i=1}^{\infty} \| x_{2i} \| t_{k_{2i}} \right\| + K \left\| \sum_{i=1}^{\infty} \| x_{2i-1} \| t_{k_{2i-1}} \right\| + 2^{-1} \left\| \sum_{k=1}^{\infty} y_k \right\|,$$

where K is a constant which depends on the degree of equivalence between $\{B_n\}_{n=1}^{\infty}$ and an "isometric" F.D.D. of type T. Hence, by condition (1) and the 1-unconditionality of $\{t_n\}_{n=1}^{\infty}$, we get

$$\left\| \sum_{i=1}^{\infty} y_i \right\| \leqq 4K \left\| \sum_{i=1}^{\infty} \| x_i \| t_{k_i} \right\| \leqq 32K \left\| \sum_{i=1}^{\infty} \| y_i \| t_{k_i} \right\|.$$

Using the version of Corollary 7 (ii) for F.D.D.'s of type T^* alluded to in Remark 2 after Corollary 12, we apply the same argument to a sequence y_i^* in D_i^*, $i = 1,2,\cdots$, which is eventually zero, and conclude that

$$\left\| \sum_{i=1}^{\infty} y_i^* \right\| \leqq 4K \left\| \sum_{i=1}^{\infty} \| y_i^* \| t_{k_i}^* \right\|.$$

Finally, by a straightforward duality argument, it follows that

$$\left\| \sum_{i=1}^{\infty} y_i \right\| \geqq (4KM)^{-1} \left\| \sum_{i=1}^{\infty} \| y_i \| t_{k_i} \right\|. \qquad \square$$

The most interesting application of the "blocking principle" proved in Theorem 13 is the following result which asserts in particular that T^* is minimal.

THEOREM 14. *Any infinite dimensional subspace of a quotient space of T^* contains in turn a subspace which is isomorphic to T^*.*

PROOF. Let X be an infinite dimensional subspace of a quotient space of T^* and let X_0 be a subspace of X with a Schauder basis $\{x_n\}_{n=1}^{\infty}$. By interpreting the basis as an F.D.D. for X_0 and by applying Theorem 13, we conclude that $\{x_n\}_{n=1}^{\infty}$

contains a subsequence which is equivalent to a subsequence $\{t^*_{k_n}\}_{n=1}^\infty$ of $\{t^*_n\}_{n=1}^\infty$. Therefore, in order to complete the proof it suffices to show that, for any increasing sequence $\{k_n\}_{n=1}^\infty$ of positive integers, T^* is isomorphic to a subspace of $[t^*_{k_n}]_{n=1}^\infty$.

Let $\{k_n\}_{n=1}^\infty$ be such an increasing sequence of integers and put $V=[t^*_{k_n}]_{n=1}^\infty$. Let $m_1=1$, $E_1=\{1,2,\cdots,k_{m_1}\}$ and $U_1=[t^*_n]_{n\in E_1}$. Since $[t^*_{k_n}]_{n=h}^{2h}$ is 2-equivalent to the unit vector basis of l_∞^h, for all h, and since l_∞ is a universal space for all separable spaces we may choose an n_1, such that if $F_1=\{k_1,k_2,\cdots,k_{n_1}\}$ and $V_1=[t^*_n]_{n\in F_1}$ then there exists an invertible operator L_1 from U_1 into V_1 with $\|L_1^{-1}\|=1$ and $\|L_1\|\leqq 2$.

Let $m_2=\max(m_1,n_1)+1$ and set $E_2=\{k_{m_1}+1,k_{m_1}+2,\cdots,k_{m_2}\}$ and $U_2=[t^*_n]_{n\in E_2}$. As in the previous case, choose an integer n_2 so that if $F_2=\{k_{n_1+1},k_{n_1+2},\cdots,k_{n_2}\}$ and $V_2=[t^*_n]_{n\in F_2}$ then there exists an operator L_2 from U_2 into V_2 with $\|L_2^{-1}\|=1$ and $\|L_2\|\leqq 2$. Continuing so, we can construct finite subsets $\{E_j\}_{j=1}^\infty$ and $\{F_j\}_{j=1}^\infty$ of the positive integers, subspaces $U_j=[t^*_n]_{n\in E_j}$ and $V_j=[t^*_n]_{n\in F_j}$ of T^*; $j=1,2,\cdots$, and invertible operators L_j from U_j into V_j; $j=1,2,\cdots$ such that

(i) $1\leqq E_1<E_2<\cdots<E_j<\cdots$ and $1\leqq F_1<F_2<\cdots<F_j<\cdots$,

(ii) $\bigcup_{j=1}^\infty E_j=\mathbf{N}$ and $\bigcup_{j=1}^\infty F_j=\{k_n\}_{n=1}^\infty$,

(iii) $F_j\subset E_j\cup E_{j+1}$, for all j,

(iv) $\|L_j^{-1}\|=1$ and $\|L_j\|\leqq 2$, for all j.

By using Theorem 8 and the remark thereafter adapted for T^* (see Remark 2 following Corollary 12), it follows that the operator $L:T^*\to V$, defined by

$$Lw=\sum_{j=1}^\infty L_j(E_jw);\qquad w\in T^*,$$

is an isomorphism from T^* onto a subspace of V. □

REMARK. The space T is not minimal. This fact, which is a consequence of the result mentioned in the Introduction that the unit vector bases in T and T_M are equivalent, is proved in [4].

The blocking principle described in Theorem 13 can be used to extend results known for c_0 or l_p-spaces to the case of spaces having an F.D.D. of type T or T^*.

PROPOSITION 15. *Let V be a Banach space having an F.D.D. of type T (respectively, T^*). Then every quotient space of V is isomorphic to a subspace of a space with an F.D.D. of type T (respectively, T^*).*

PROOF. If the quotient of V has an F.D.D., Proposition 15 is a special case of

Theorem 13. The general case is deduced from the Theorem 13 case in the same way as the corresponding statement in l_p (cf. theorem 1 in [9]). □

It is well known that quotient spaces of c_0 are isomorphic to subspaces of c_0 (cf. [9]). As an application of Proposition 15, we prove that a similar assertion holds for T^*.

THEOREM 16. *Every quotient space of T^* is isomorphic to a subspace of T^* and, hence, every subspace of T is isomorphic to a quotient space of T.*

PROOF. Let W be a quotient space of T^*. By Proposition 15, W is isomorphic to a subspace of a space Z having a F.D.D. of type T^*. The proof will be completed once we show that Z itself embeds isomorphically in T^*.

Suppose that $\{Z_n\}_{n=1}^\infty$ is an F.D.D. of type T^* for Z relative to a sequence $\{k_n\}_{n=1}^\infty$. Choose now an integer $m(1)$ so that $W_0 = Z_1$ is 2-isomorphic to a subspace of $[t_{k_n}^*]_{n=1}^{m(1)}$ (use the fact that, for each m, $\{t_{k_n}^*\}_{n=m}^{2m}$ is 2-equivalent to the unit vector basis in l_∞^m). Consider now $W_1 = [Z_n]_{n=2}^{m(1)}$ and choose an integer $m(2) > m(1)$ so that W_1 is 2-isomorphic to a subspace of $[t_{k_n}^*]_{n=m(1)+1}^{m(2)}$. Continuing so, we construct a blocking $\{W_i\}_{i=0}^\infty$ of $\{Z_n\}_{n=1}^\infty$ and an increasing sequence $\{m(i)\}_{i=0}^\infty$ of integers (where $m(0)=0$) so that, for each $i > 2$, $W_i = [Z_n]_{n=m(i-1)+1}^{m(i)}$ is 2-isomorphic to a subspace of $[t_{k_n}^*]_{n=m(i)+1}^{m(i+1)}$.

As mentioned above, $\{W_i\}_{i=0}^\infty$ is also an F.D. of type T^* for Z but relative to $\{k_{m(i)}\}_{i=1}^\infty$. Finally, for each i, let L_i be an isomorphism from W_i onto a subspace of $[t_{k_n}^*]_{n=m(i)+1}^{m(i+1)}$ so that $\|L_i^{-1}\| = 1$ and $\|L_i\| \leqq 2$. By the version of Theorem 8 for F.D.D.'s of type T^*, the operator $L : Z \to [t_{k_n}^*]_{n=1}^\infty \subset T^*$ defined by: $Lz = \sum_{i=0}^\infty L_i w_i$, for $z = \sum_{i=0}^\infty w_i \in Z$ with $w_i \in W_i$, $i = 1, 2, \cdots$, is an isomorphism of Z onto a subspace of T^*. □

REFERENCES

1. S. F. Bellenot, *The Banach space T and the fast growing hierarchy from logic*, preprint.

2. J. Bourgain, P. G. Casazza, J. Lindenstrauss and L. Tzafriri, *Banach spaces with a unique unconditional basis, up to permutation*, in preparation.

3. P. G. Casazza, *Tsirelson's space*, Proc. Research Workshop on Banach Space Theory, University of Iowa (Bor-Luh Lin, ed.), 1981, pp. 9–22.

4. P. G. Casazza and E. W. Odell, *Tsirelson's space and minimal subspaces*, Longhorn Notes, University of Texas, 1982–83, pp. 61–72.

5. T. Figiel and W. B. Johnson, *A uniformly convex Banach space which contains no l_p*, Compos. Math. **29** (1974), 179–190.

6. W. B. Johnson, *A reflexive Banach space which is not sufficiently Euclidean*, Studia Math. **55** (1976), 201–205.

7. W. B. Johnson, *On quotients of L_p which are quotients of l_p*, Compos. Math. **34** (1977), 69–89.

8. W. B. Johnson and M. Zippin, *On subspaces and quotients of $(\Sigma G_n)_{l_p}$ and $(\Sigma G_n)_{c_0}$*, Isr. J. Math. **13** (1972), 311–316.

9. W. B. Johnson and M. Zippin, *Subspaces and quotient spaces of* $(\sum G_n)_{l_p}$ *and* $(\sum G_n)_{c_0}$, Isr. J. Math. **17** (1974), 50–55.

10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Ergebnisse 92, Springer-Verlag, 1977.

11. B. S. Tsirelson, *Not every Banach space contains* l_p *or* c_0, Funct. Anal. Appl. **8** (1974), 138–141.

THE UNIVERSITY OF MISSOURI AND THE UNIVERSITY OF ALABAMA

THE OHIO STATE UNIVERSITY AND TEXAS A&M UNIVERSITY

THE HEBREW UNIVERSITY OF JERUSALEM, THE UNIVERSITY OF CAMBRIDGE, AND THE OHIO STATE UNIVERSITY